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Albert-Ludwigs-Universität Freiburg
Sommersemester 2024

Topology

Problem Sheet 3

Deadline: 07 May 2024, 15h

Exercise 1 (4 Points)**.**

An *open covering* of a topological space X is a family $(U_i)_{i\in I}$ of open subsets of X such that $X = \bigcup_{i\in I} U_i$. Show that the following statements are equivalent for a given subset Z of X:

a) The set Z is closed in X.

b) For every open covering $(U_i)_{i\in I}$ of X, the set $Z \cap U_i$ is closed in U_i with respect to the subspace topology on U_i.

c) There is some open covering $(V_j)_{j\in J}$ of X such that the set $Z \cap V_j$ is closed in V_j with respect to the subspace topology on V_j.

Exercise 2 (4 Points)**.**

Define the following equivalence relation in the Euclidean line $\mathbb{R}$:

$$xEy \Longleftrightarrow x \cdot y > 0 \text{ or } x = y = 0.$$

a) How many equivalence classes are there?

b) Completely describe all open sets of the quotient topology with respect to this equivalence relation. Which classes in the quotient space are isolated?

c) Is the quotient space R/E a T_1 space?

Exercise 3 (12 Points)**.**

In a topological space X let E be an equivalence relation and consider Δ_E be the set of all pairs (x, y) in $X \times X$ such that $E(x, y)$.

a) Assume that the quotient map $p_E : X \to X/E$ is open. Show that Δ_E is closed with respect to the product topology on $X \times X$ if and only if the quotient space X/E is Hausdorff.

From now on, the topological space X will be a *a topological group*, that is, there is a group law on X such that both the inversion map

$$\begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

and the multiplication map

$$\begin{array}{ccc} X \times X & \to & X \\ (x, y) & \mapsto & x \cdot y \end{array}$$

are continuous, where $X \times X$ is equipped with the product topology.

b) Show that the inversion map is a homeomorphism.

c) Show the following equivalences: X is $T_2 \Leftrightarrow X$ is $T_1 \Leftrightarrow$ the trivial subgroup $\{e\}$ is closed.

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Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im entsprechenden Fach im Keller des mathematischen Instituts.

d) Show that the subgroup H of X is closed if and only if the subset

$$\{(x, y) \in X \times X \mid y^{-1} \cdot x \in H\}$$

is closed in the product topology.

e) Show that the subgroup H is closed whenever H is open in X.

f) Show that for any subgroup H of X, the quotient map $X \to X/H$ is always open, where X/H is equipped with the quotient topology.

g) Conclude that the quotient group X/H is Hausdorff if X is Hausdorff and H is a closed subgroup.